A Spatial Clustering Algorithm for Sparsifying Reduced Order Models
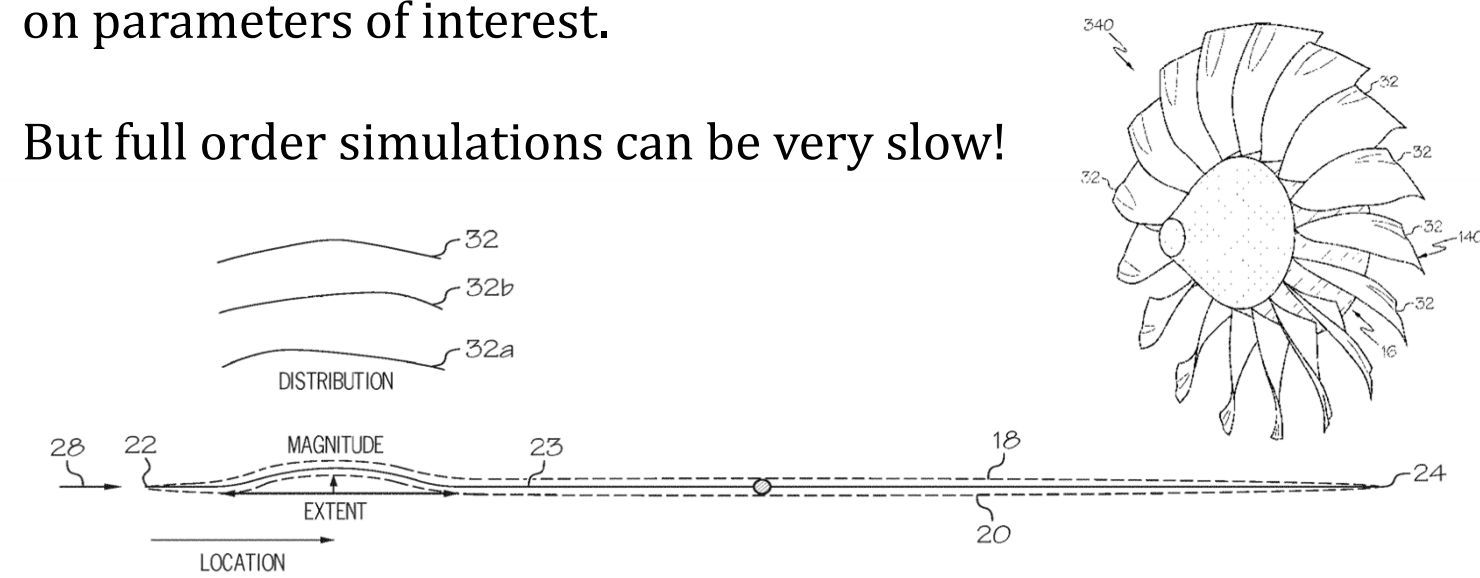
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The Problem

Numerical experiments can tell you how a parameter will influence your design's mechanical or aerodynamic performance. For example, the local changes to fan blade curvature shown may have a large influence on the flow and on parameters of interest.

But full order simulations can be very slow!

Reduced Order Models

ROMs limit the degrees of freedom in a numerical experiment in order to run faster simulations using knowledge of the problem and knowledge gained from previous simulations.
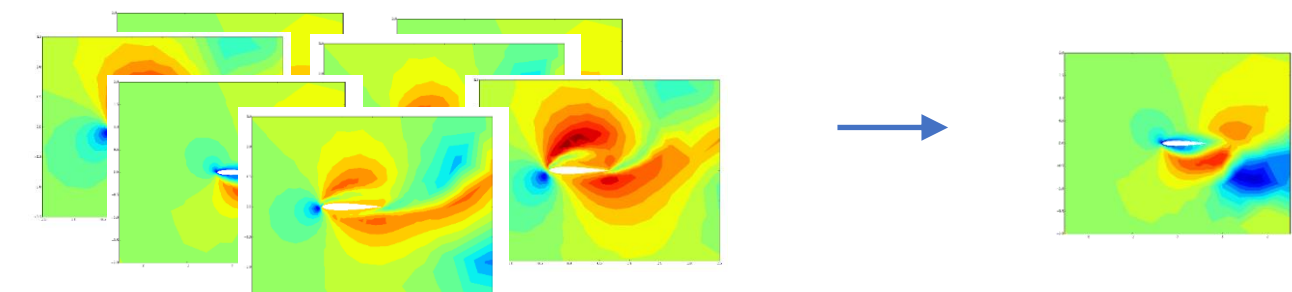
ROMs solve the original governing equations, but project them into a smaller state space, called a basis, reducing simulation time possibly by several orders of magnitude.

Start with a PDE $\qquad \dot{w} + F(w) = 0$

Discretize $\qquad \dfrac{w^{n+1} - w^n}{\Delta t} + F(w^n) = 0$

$$w^{n+1} = w^n + \Delta t\, F(w^n)$$

How do you pick a basis V?

Reduce dimensionality $\qquad w \sim \boldsymbol{V}y$

$w \in \mathbb{R}^M,\ V \in \mathbb{R}^{M \times n},\ y \in \mathbb{R}^n \qquad y^{n+1} = y^n + \Delta t\, \boldsymbol{V}^T F(\boldsymbol{V}y^n)$
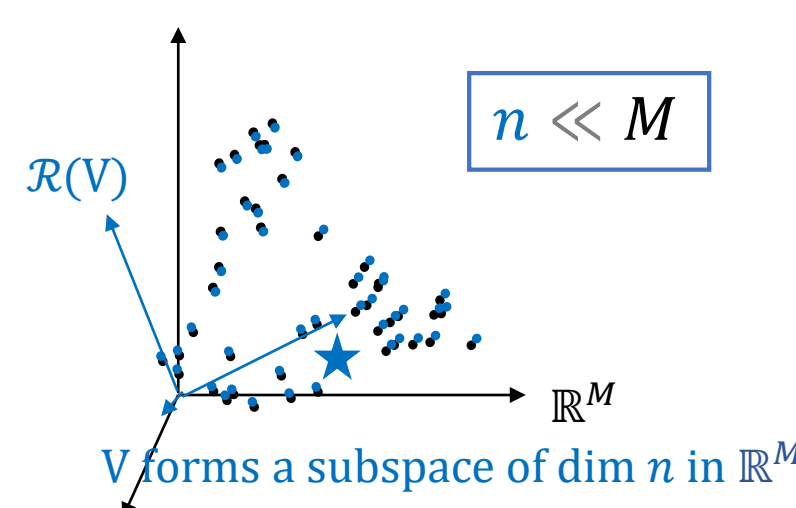
What is a Basis?

Given a snapshot matrix A of with full order model results stored in the columns, a basis V is constructed.

$A \in \mathbb{R}^{M \times N}$

$A = U\Sigma V^T$

$\boldsymbol{V} = U_{truncated}$

$n \ll M$

Perform SVD . Truncate and save first n left singular vectors $V \in \mathbb{R}^{M \times n}$ to capture the desired energy content.

$\mathcal{R}(V)$

V forms a subspace of dim n in $\mathbb{R}^M$

Column Clustered Basis

Similar snapshots are grouped together using the k-means algorithm. Within each column cluster, the ROM solves within a basis built only from the simulations most similar to the current state, further reducing the dimensionality. Matrix A columns correspond to states.
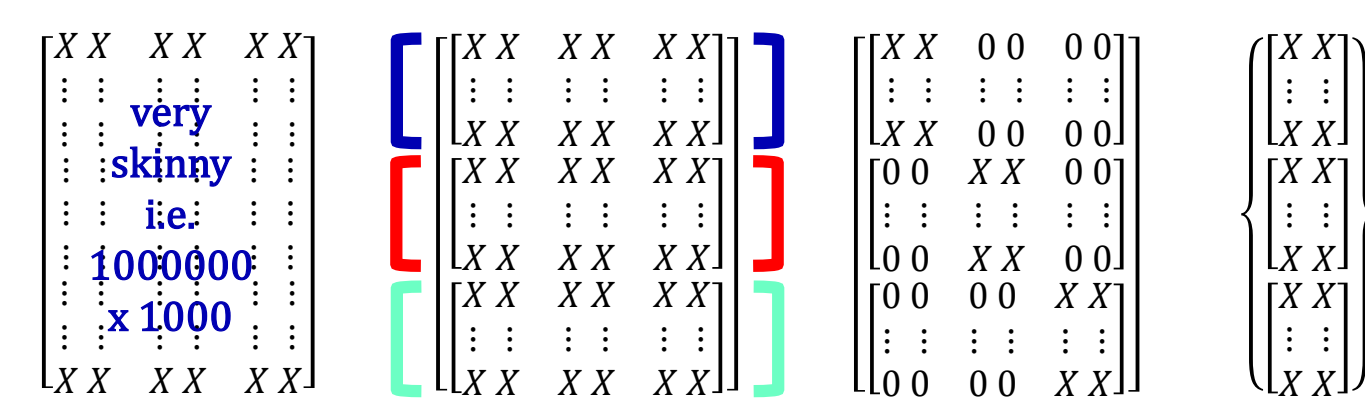

A =

$n_1 \ll M$
$n_2 \ll M$
$n_3 \ll M$

$\mathbb{R}^M$

$n_1 + n_2 + n_3 \leq N$
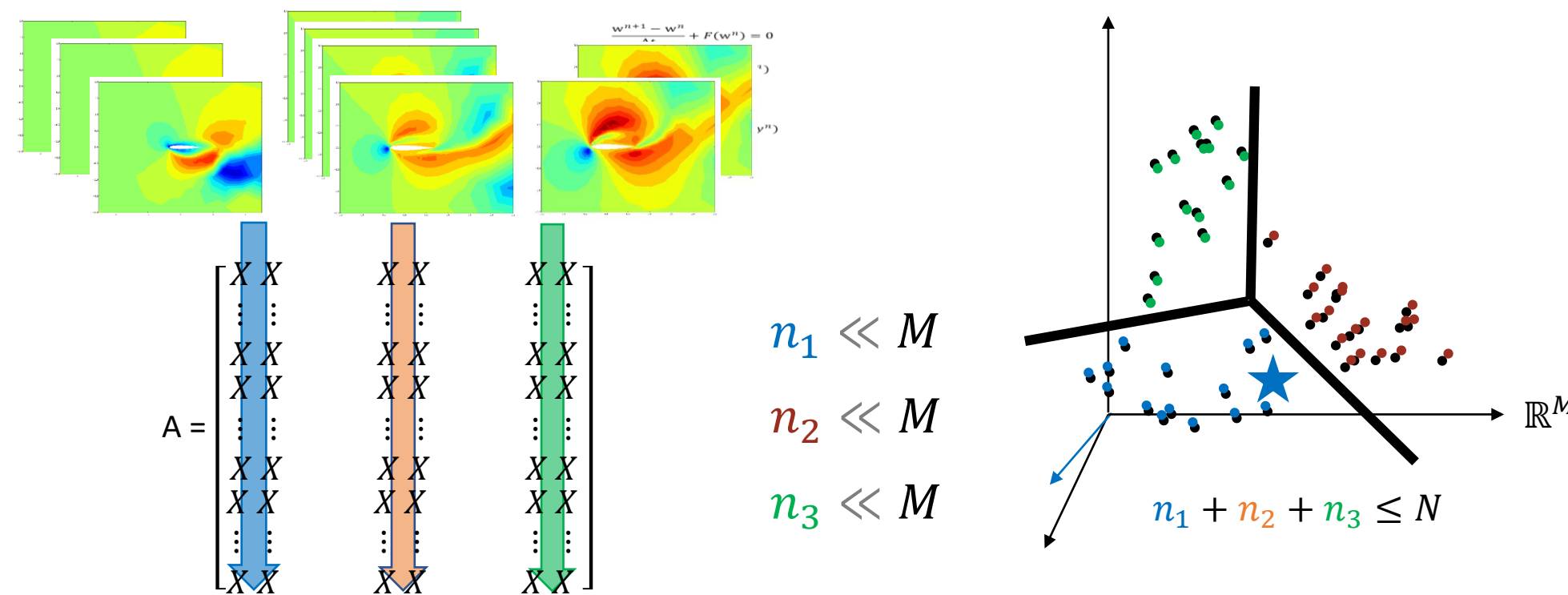
– New Method –
Row Clustered Basis

Points in space, corresponding to a rows of the matrix A, can also be clustered, separating spatial regions that show similar behavior over time – roughly identifying flow features.

$M_1 + M_2 + M_3 = M$

$\mathbb{R}^N$
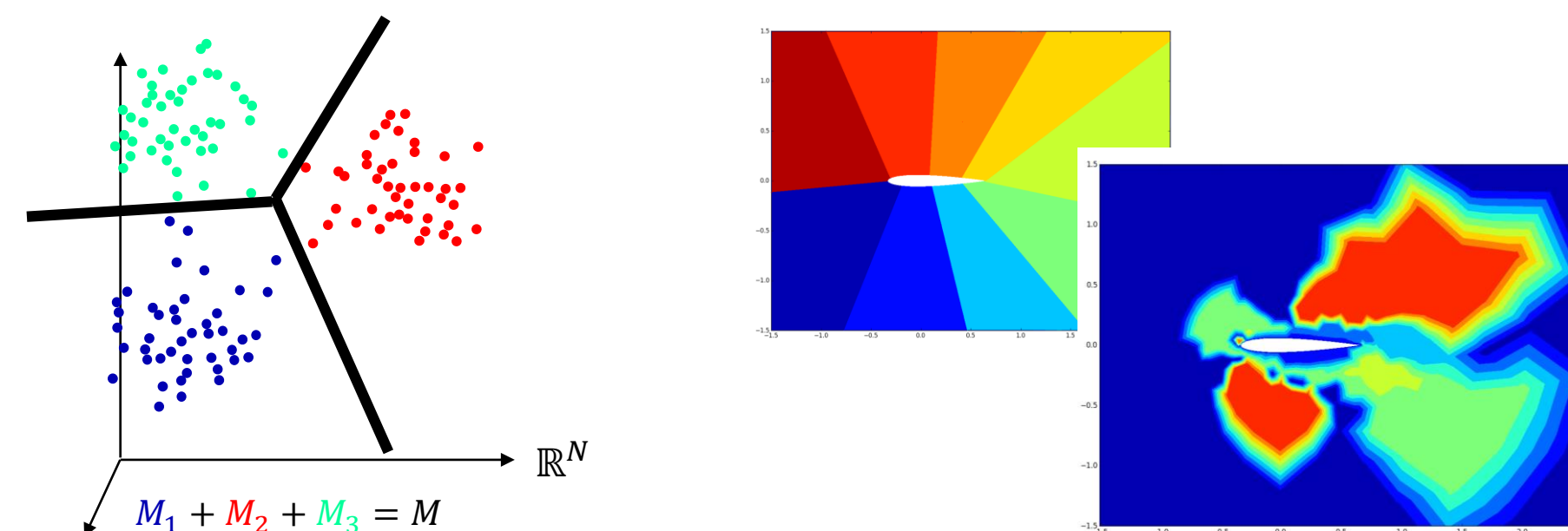
<u>Expected Benefit #1: Spatial Locality.</u> Different spatial flow features are identified and modeled, improving accuracy/efficiency for spatially varying phenomena (i.e. steady-state shocks).

very skinny i.e. 1000000 x 1000

① cluster ② SVD ③ store ④

③ ④

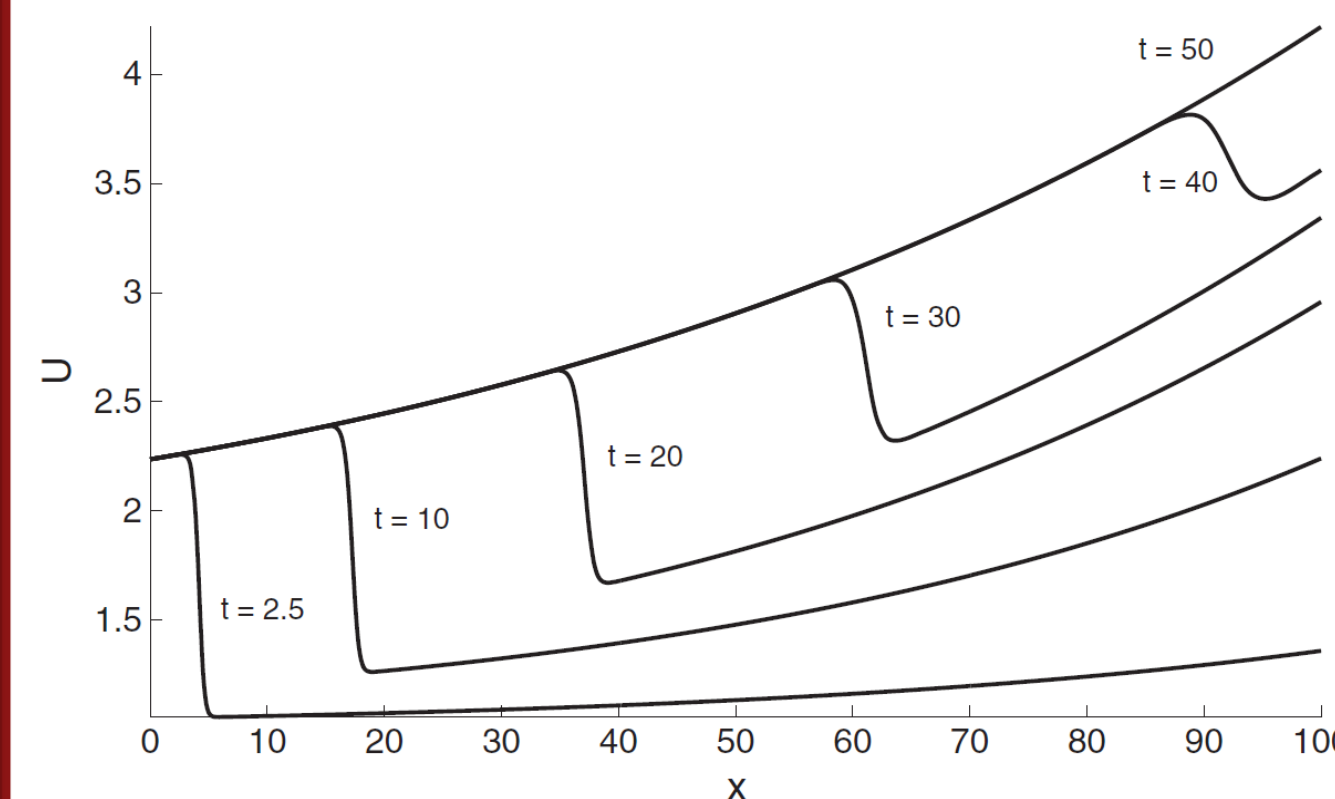still very skinny i.e. 1000000 x 3000

<u>Expected Benefit #2: Sparsification.</u> Can choose any option in between (Scenario 1) Faster computation time for a fixed accuracy and using the same basis size and (Scenario 2) Greater accuracy for a fixed computation time using an increased basis size.

Scenario 1

Use the same number of basis vectors with a same-sized sparser matrix while storing fewer nonzero values, **similar** accuracy with **faster** computation time.

Scenario 2

Use more basis vectors with a larger sparser matrix while storing the same number of non-zero values, **greater** accuracy with **similar** computation time.

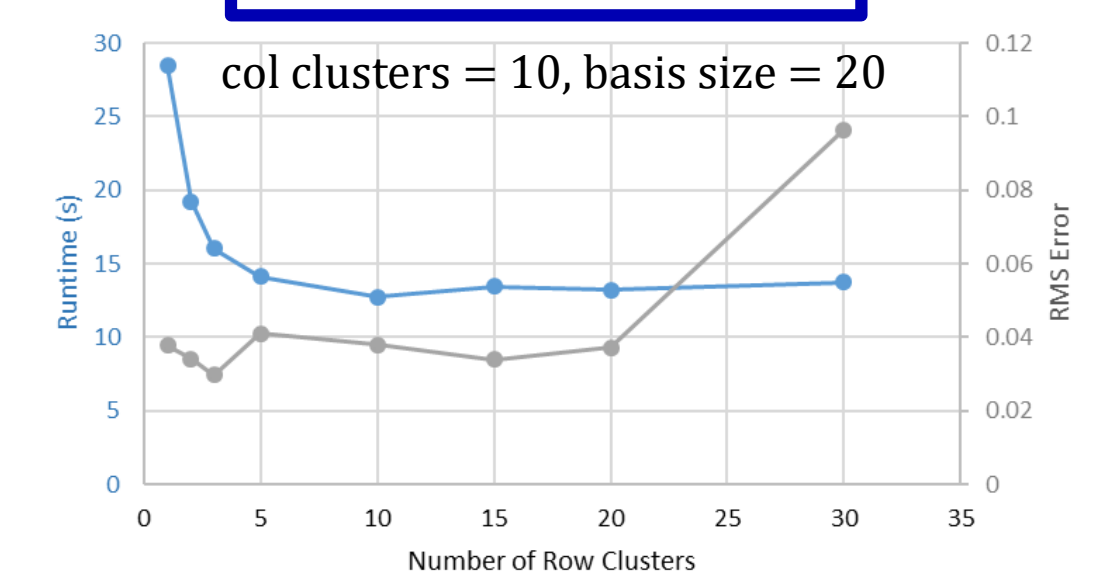## Results – Row + Column Clustered Basis

The Burgers' equation test case for ROMs is a 1D application of an initial-boundary-value problem that models the movement of a shockwave in an inviscid fluid.

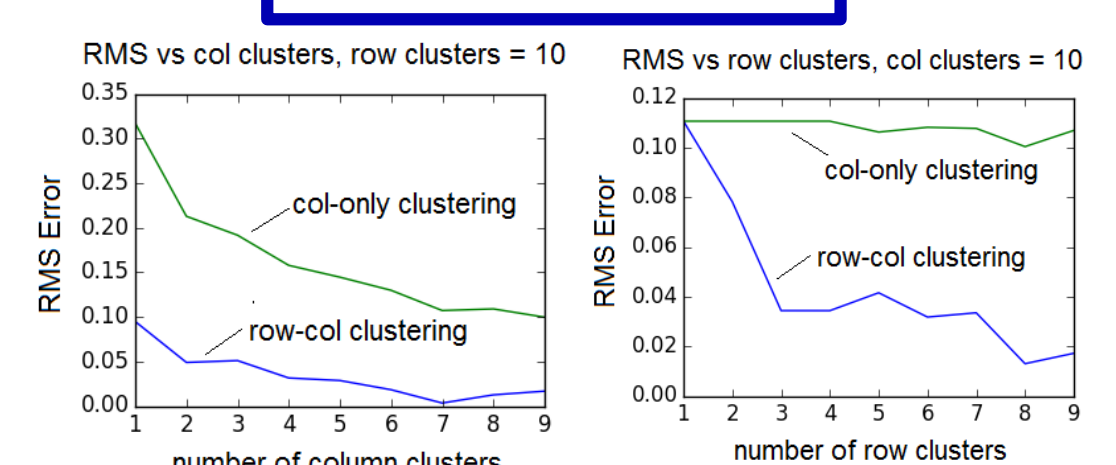$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = 0.02e^{0.02x}$$

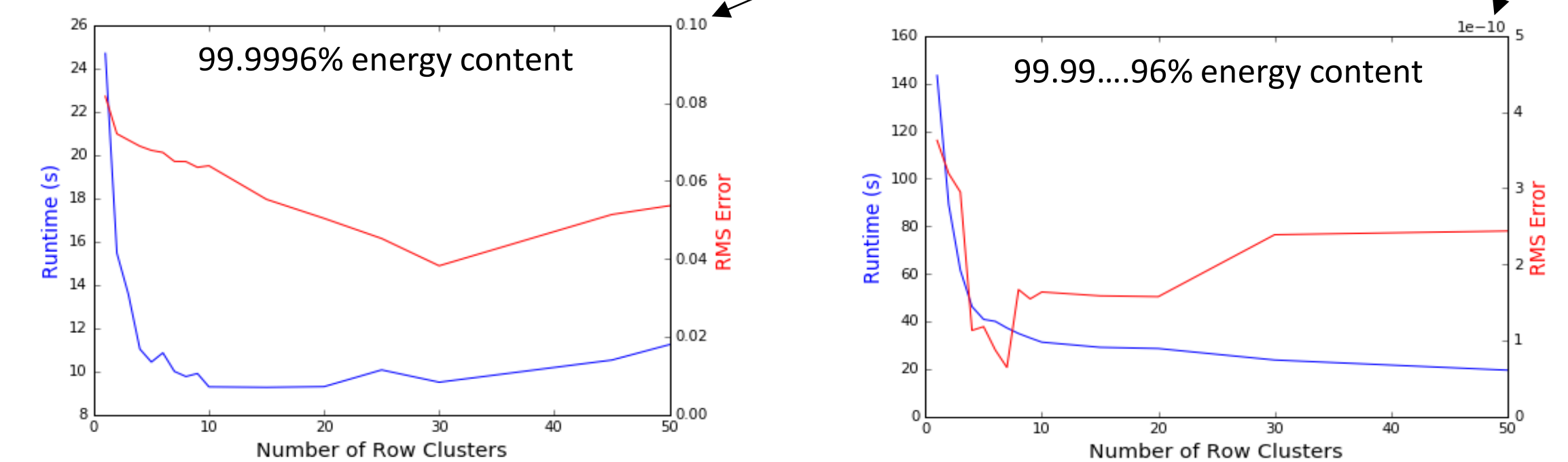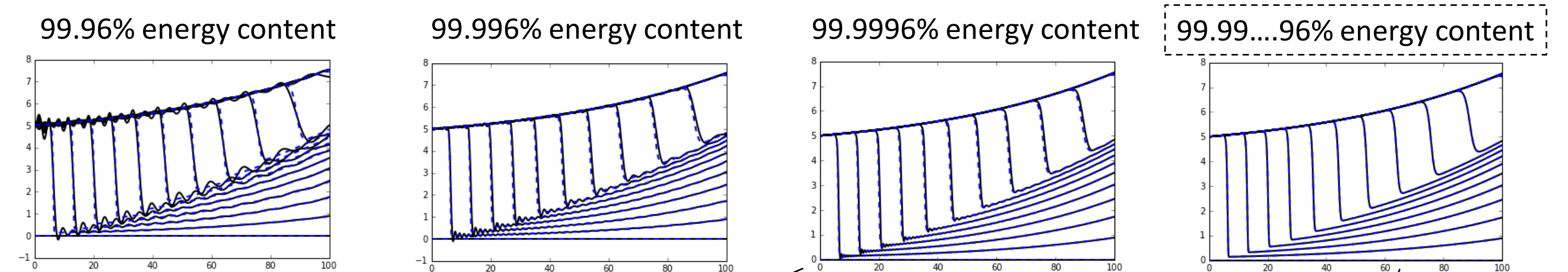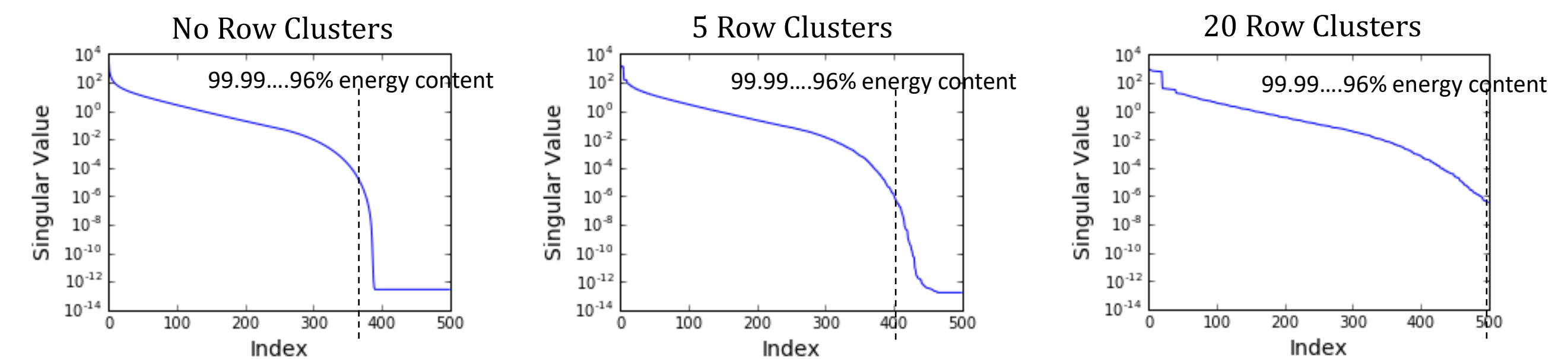$U(x,0) = 1 \quad U(0,t) = \sqrt{5} \quad x \in [0,100] \quad t \in [0,50]$

Scenario 1
col clusters = 10, basis size = 20

Scenario 2
RMS vs col clusters, row clusters = 10
RMS vs row clusters, col clusters = 10

col-only clustering
row-col clustering

col-only clustering
row-col clustering

Results - Row Only Clustered Basis

The most relevant characteristic in a ROMs is the amount of energy contained in the basis. Typically the size of the basis is chosen by truncating at the number of vectors for which the singular values begin to drop quickly.

No Row Clusters — 99.99....96% energy content
5 Row Clusters — 99.99....96% energy content
20 Row Clusters — 99.99....96% energy content

99.96% energy content
99.996% energy content
99.9996% energy content
99.99....96% energy content

99.9996% energy content
99.99....96% energy content

For any chosen energy content, the RMS error of the ROM remains constant while the ROM runtime is reduced by a factor of 3 to 7. A variation of the method can be used to hold the runtime constant while reducing RMS error.